Fermion loops, loop cancellation and density correlations in two dimensional Fermi systems

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February 7, 2008

Abstract

We derive explicit results from an exact expression for fermion loops with an arbitrary number of density vertices in two dimensions at zero temperature, which has been obtained recently by Feldman et al. [1]. The 3-loop is an elementary function of the three external momenta and frequencies, and the N-loop can be expressed as a linear combination of 3-loops with coefficients that are rational functions of momenta and frequencies. We show that the divergencies of single loops for low energy and small momenta cancel each other when loops with permuted external variables are summed. The symmetrized N-loop, i.e. the connected N-point density correlation function of the Fermi gas, does not diverge for low energies and small momenta. In the dynamical limit, where momenta scale to zero at fixed finite energy variables, the symmetrized N-loop vanishes as the (2N−2)-th power of the scale parameter.

PACS: 71.10.-w, 71.10.Ca, 71.10.Pm

1. Introduction

The discovery of high temperature superconductivity and the peculiar phenomena observed in two-dimensional electron gases have stimulated much interest in the low energy physics of interacting Fermi systems with singular interactions and reduced dimensionality. In particular, the possible breakdown of Fermi liquid theory in such systems has been a challenging issue [2].

In this context the properties of fermion loops (see Fig. 1) with density or current vertices play a very important role. Such loops appear as subdiagrams of Feynman diagrams and also as kernels in effective actions for collective degrees of freedom. Their behavior for small external momenta and energies determines many properties of the whole system. A single loop with N vertices diverges in the small energy-momentum limit for $N > 2$. In the symmetrized loop, obtained by summing all permutations of the external energy-momentum variables $q_1, \ldots, q_N$, at least the leading contributions from single loops in the small energy-momentum limit are known to cancel each other [3, 2, 4]. This systematic cancellation is known as *loop cancellation*. As a consequence, loops with $N > 2$ are often irrelevant for the low-energy behavior, and can therefore be neglected in effective actions or in resummations of the perturbation series via Ward identities or bosonization [2, 4]. This implies that collective density fluctuations and the associate response functions can be described by a renormalized random phase approximation, as in a Fermi liquid, even if interactions destroy Landau quasi-particles.

While the 2-loop, known as polarization insertion or bubble diagram, has been computed long ago in one, two and three dimensions, no explicit formulae are available for higher order loops. Furthermore, the available proofs of loop cancellation show only that some cancellation occurs, but they do not show by how many powers the divergence is reduced. To be able to carry out an accurate power-counting of the infrared divergencies of systems with singular interactions or soft bosonic modes, a more detailed knowledge of the behavior of loops is required.

Recently, Feldman, Knörrer, Sinclair and Trubowitz [1] have obtained an exact expression for the N-loop with density vertices in a two dimensional Fermi gas at zero temperature. In this article we derive explicit formulae for loops with density vertices in two dimensions from their expression, which are useful for the evaluation of Feynman diagrams containing such loops as subdiagrams. We show that loop cancellation reduces the degree of divergence by $N-2$ powers for $N > 2$, i.e. the symmetrized loops, which are equal to the N-point density correlation functions of the Fermi gas, are generically *finite* in the small energy-momentum limit. In the dynamical limit, where the momenta scale

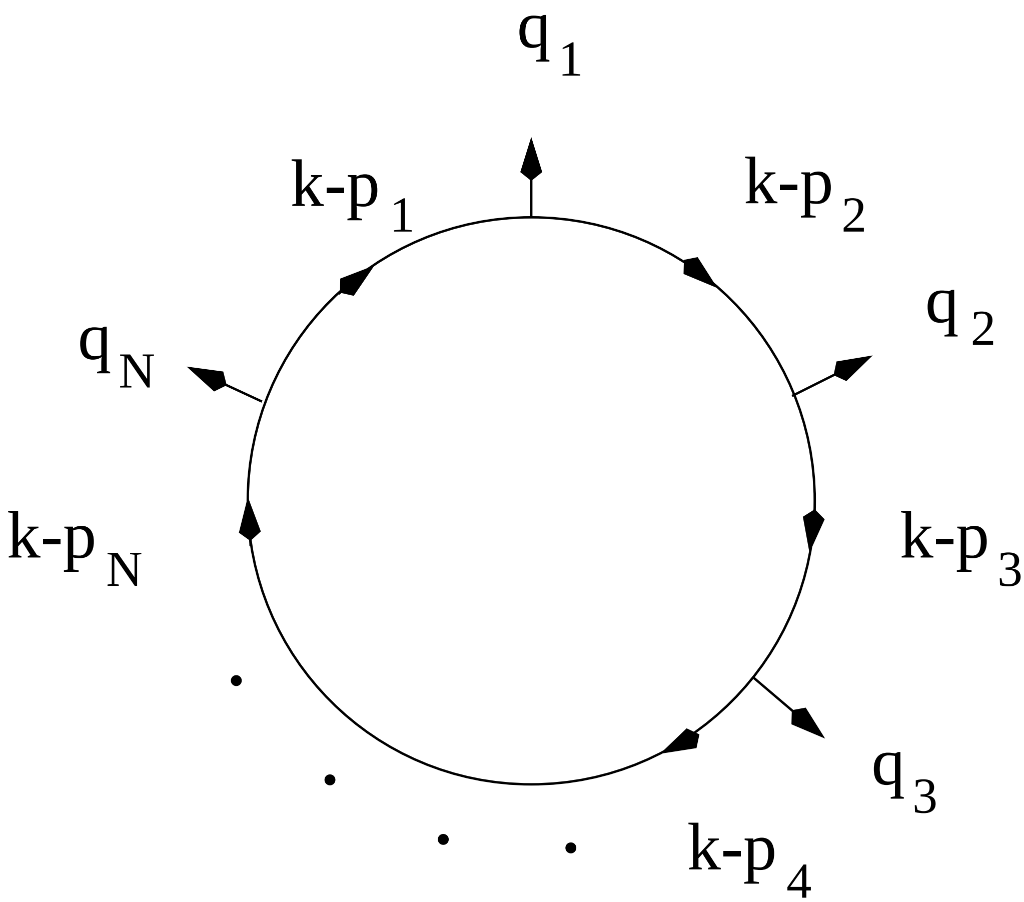

Figure 1: The N-loop with its energy-momentum labels.

to zero at fixed finite energy variables, the symmetrized N-loop vanishes as the (2N−2)-th power of the scale parameter.

This article is organized as follows. In Sec. 2 we define single and symmetrized loops and review briefly known basic properties. In Sec. 3 we summarize the main results on loops in two dimensions obtained by Feldman et al. [1], which are the starting point for our analysis. In Sec. 4 the 3-loop is evaluated explicitly, and its asymptotic behavior for small energy and momentum variables is analysed. In Sec. 5 we derive a theorem clarifying the structure of a certain symmetrized product, which allows us to control the asymptotic low energy and small momentum behavior of symmetrized N-loops, to be discussed in Sec. 6. A conclusion follows in Sec. 7.

2. Definitions and simple properties of loops

The amplitude of the N-loop with density vertices, represented by the Feynman dia-

gram in Fig. 1, is given by

$$\Pi_{\mathrm{N}}(q_1,\ldots,q_{\mathrm{N}}) = I_{\mathrm{N}}(p_1,\ldots,p_{\mathrm{N}}) = \int \frac{d^dk}{(2\pi)^d} \int \frac{dk_0}{2\pi} \prod_{j=1}^{\mathrm{N}} G_0(k-p_j) \tag{1}$$

at temperature zero. Here $k = (k_0, \mathbf{k})$, $q_j = (q_{j0}, \mathbf{q}_j)$ and $p_j = (p_{j0}, \mathbf{p}_j)$ are energy-momentum vectors. The variables q_j and p_j are related by the linear transformation

$$q_j = p_{j+1} - p_j \ , \quad j = 1,\ldots,\mathrm{N} \tag{2}$$

where $p_{\mathrm{N}+1} \equiv p_1$. Energy and momentum conservation at all vertices yields the restriction $q_1 + \ldots + q_{\mathrm{N}} = 0$. The variables $q_1,\ldots,q_{\mathrm{N}}$ fix $p_1,\ldots,p_{\mathrm{N}}$ only up to a constant shift $p_j \mapsto p_j + p$. Setting $p_1 = 0$, one gets

$$\begin{aligned} p_2 &= q_1 \\ p_3 &= q_1 + q_2 \\ &\vdots \\ p_{\mathrm{N}} &= q_1 + q_2 + \ldots + q_{\mathrm{N}-1} \end{aligned} \tag{3}$$

We use the imaginary time representation, with a non-interacting propagator

$$G_0(k) = \frac{1}{ik_0 - (\epsilon_{\mathbf{k}} - \mu)} \tag{4}$$

where $\epsilon_{\mathbf{k}}$ is the dispersion relation and μ the chemical potential of the system. The k_0-integral in Eq. (1) can be easily carried out using the residue theorem; one obtains [1]

$$I_{\mathrm{N}}(p_1,\ldots,p_n) = \sum_{i=1}^{N} \int_{|\mathbf{k}-\mathbf{p}_i|<k_F} \frac{d^dk}{(2\pi)^d} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} f_{ij}(\mathbf{k}) \right)^{-1} \tag{5}$$

where $f_{ij}(\mathbf{k}) = \epsilon_{\mathbf{k}-\mathbf{p}_i} - \epsilon_{\mathbf{k}-\mathbf{p}_j} + i(p_{i0} - p_{j0})$.

The 2-loop $\Pi_2(q,-q) \equiv \Pi(q)$ is known as polarization insertion or particle-hole bubble, and has a direct physical meaning: $\Pi(q)$ is the dynamical density-density correlation function of a non-interacting Fermi system [5]. While the 2-loop has been computed explicitly for continuum systems in one, two and three dimensions already long ago [2], only few explicit results exist for higher loops.

The behavior of loops for small q_j is particularly important, because interactions and bosonic propagators attached to the fermion loops in general Feynman diagrams are often singular for small energies and momenta [2].

A simple formula for the *static* small-q limit has been derived long ago by Hertz and Klenin [6], namely

$$\lim_{\mathbf{q}_j \to 0} \lim_{q_{j0} \to 0} \Pi_{\mathrm{N}}(q_1, \ldots, q_{\mathrm{N}}) = \frac{(-1)^{\mathrm{N}-1}}{(\mathrm{N}-1)!} \left. \frac{d^{\mathrm{N}-2} D(\epsilon)}{d\epsilon^{\mathrm{N}-2}} \right|_{\epsilon=\mu} \tag{6}$$

where $D(\epsilon) = \int \frac{d^d k}{(2\pi)^d} \delta(\epsilon - \epsilon_{\mathbf{k}})$ is the density of states. Note that the limit $q_j \to 0$ is not unique; the above result is valid only in the so-called static limit where the energy variables go to zero first.

In the following we will analyze the behavior of loops in the generic small-q limit, $\lim_{\lambda \to 0} \Pi_{\mathrm{N}}(\lambda q_1, \ldots, \lambda q_{\mathrm{N}})$ for arbitrary $q_1, \ldots, q_{\mathrm{N}}$, with a vanishing scaling factor λ applied to all energy and momentum variables. According to a simple power-counting estimate of the integral in Eq. (1) one would expect that an N-loop with $N > 2$ diverges as λ^{2-N} for $\lambda \to 0$, since for $q_j = 0$ one integrates over an N-fold divergence on the $(d-1)$-dimensional manifold defined by $k_0 = 0$, $\epsilon_{\mathbf{k}} = \mu$, which has codimension two in a $(d+1)$-dimensional energy-momentum space. This estimate yields however only an upper bound for the degree of divergence, since the actual value of the integral may be smaller due to cancellations of contributions with opposite signs, as is obviously the case in the static limit (6).

We will also analyze the so-called *dynamical* limit, where a vanishing scaling factor λ is applied only to momentum variables, at fixed finite energy variables.

Divergencies of single loops in the small-q limit cancel each other at least to some degree in the *symmetrized* loop

$$\Pi^S_{\mathrm{N}}(q_1, \ldots, q_{\mathrm{N}}) = \mathcal{S}\, \Pi_{\mathrm{N}}(q_1, \ldots, q_{\mathrm{N}}) = \frac{1}{\mathrm{N}!} \sum_P \Pi_{\mathrm{N}}(q_{P1}, \ldots, q_{P\mathrm{N}}) \tag{7}$$

where the symmetrization operator $\mathcal{S}$ imposes summation over all permutations of $q_1, \ldots,$ q_{N}. The behavior of single loops is physically not relevant, since all physical properties can be expressed in terms of symmetrized loops. This important point is however ignored in some approximation schemes. The symmetrized N-loop is proportional to the N-point density correlation function of a non-interacting Fermi system with dispersion relation $\epsilon_{\mathbf{k}}$ and chemical potential μ. In the one-dimensional Luttinger model [7], which has a linear dispersion relation, loops with $N > 2$ cancel completely, i.e. $\Pi^S_{\mathrm{N}} \equiv 0$, as first noticed by Dzyaloshinskii and Larkin [8]. In other systems the loop cancellation is not complete and, in general, Π^S_{N} does not vanish in the small-q limit. Only in the dynamical limit Ward identities associated with particle number conservation imply that Π^S_N goes to zero at least linearly in each momentum variable [9, 10]. This is crucial in particular for the infrared renormalizability of the Coulomb gas [10], since the small-q singularity of the interaction

is not removed by screening in the dynamical limit. For the general small-q limit with finite ratios of momenta and energies, only the cancellation of leading singularities has already been established [3, 2, 4], but to our knowledge the behavior of the remainder has not yet been analyzed.

3. Two dimensions and the results by Feldman et al.

We now focus on *two-dimensional* Fermi systems with a quadratic dispersion relation $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$, where m is the mass of the particles. The Fermi surface is then a circle with radius $k_F = \sqrt{2m\mu}$. In the following we choose energy and momentum units such that $k_F = 1$ and $m = 1$. It is easy to reinsert arbitrary values for k_F and m by making a simple dimensional analysis of the expressions.

The 2-loop in two dimensions, first computed by Stern [11], is given by

$$\Pi(q) = \frac{1}{2\pi}\left\{-1 + \frac{1}{|\mathbf{q}|}\left[\sqrt{\left(-\frac{iq_0}{|\mathbf{q}|} + \frac{|\mathbf{q}|}{2}\right)^2 - 1} + \sqrt{\left(\frac{iq_0}{|\mathbf{q}|} + \frac{|\mathbf{q}|}{2}\right)^2 - 1}\right]\right\} \tag{8}$$

where the complex square-root is defined to have positiv real part. In the small-q limit one obtains

$$\lim_{\lambda\to 0} \Pi(\lambda q) = \frac{1}{2\pi}\left(-1 + \frac{1}{\sqrt{1 + |\mathbf{q}|^2/q_0^2}}\right) \tag{9}$$

i.e. a finite result depending on the ratio $|\mathbf{q}|/q_0$. In the dynamical limit, $\mathbf{q} \to 0$ at finite q_0, the 2-loop vanishes as $|\mathbf{q}|^2$.

Using methods from complex analysis, Feldman et al. [1] have recently reduced the problem of computing arbitrary N-loops in two dimensions to elementary integrals. We now report their results, which are the starting point for our further explicit evaluations.

If the three vectors $\mathbf{p}_i$, $\mathbf{p}_j$ and $\mathbf{p}_k$ are not collinear, one can define the two-dimensional complex vector $\mathbf{d}^{ijk}$ as the unique solution of the equations $f_{ij}(\mathbf{k}) = f_{jk}(\mathbf{k}) = 0$, where

$$f_{ij}(\mathbf{k}) = (\mathbf{p}_j - \mathbf{p}_i)\cdot\mathbf{k} + \frac{1}{2}(\mathbf{p}_i^2 - \mathbf{p}_j^2) + i(p_{i0} - p_{j0}) \tag{10}$$

Since $f_{ij} + f_{jk} + f_{ki} \equiv 0$, the vector $\mathbf{d}^{ijk}$ is also a solution of the equation $f_{ki}(\mathbf{k}) = 0$. Hence, $\mathbf{d}^{ijk}$ does not depend on the order of the indices i, j, k. The real part of $\mathbf{d}^{ijk}$ is the center of the circle circumscribing the triangle with vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$.

For the 3-loop, Feldman et al. [1] have obtained the expression

$$I_3(p_1, p_2, p_3) = \frac{1}{2\pi i \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \sum_{\substack{i,j=1\\ i\neq j}}^{3} s_{ij} \int_{\gamma_{ij}} \frac{dz}{z} \tag{11}$$

where $s_{12} = s_{23} = s_{31} = 1$, $s_{21} = s_{32} = s_{13} = -1$, and $\det(\mathbf{u}, \mathbf{v}) \equiv u_x v_y - u_y v_x$ for arbitrary two-dimensional vectors $\mathbf{u}, \mathbf{v}$. The contour-integrals are performed along the curves $\gamma_{ij} = \{w_{ij}(s) | 0 \leq s \leq 1\}$ where $w_{ij}(s)$ is the unique (generally complex) root of the quadratic equation

$$(\mathbf{p}_j - \mathbf{p}_i)^2 \, z^2 + 2 \det(\mathbf{d} - \mathbf{p}_i, \mathbf{p}_j - \mathbf{p}_i) \, z + (\mathbf{d} - \mathbf{p}_i)^2 = s^2 \tag{12}$$

satisfying the condition

$$\mathrm{Im} \, \frac{-(p_{jx} - p_{ix}) w_{ij}(s) + d_y - p_{iy}}{(p_{jy} - p_{iy}) w_{ij}(s) + d_x - p_{ix}} \; > \; 0 \tag{13}$$

with $\mathbf{d} = (d_x, d_y) = \mathbf{d}^{123}$.

The general N-loop can also be written as a complex contour integral over rational functions [1]. That expression implies that the N-loop can be expressed in terms of 3-loops as

$$I_{\mathrm{N}}(p_1, \dots, p_{\mathrm{N}}) = \sum_{1 \leq i < j < k \leq \mathrm{N}} \left[\prod_{\substack{\nu=1 \\ \nu \neq i,j,k}}^{\mathrm{N}} f_{i\nu}(\mathbf{d}^{ijk}) \right]^{-1} I_3(p_i, p_j, p_k) \tag{14}$$

This *reduction formula* follows also directly from the identity

$$\prod_{i=1}^{\mathrm{N}} G_0(k - p_i) = \sum_{1 \leq i < j < k \leq \mathrm{N}} \left[\prod_{\substack{\nu=1 \\ \nu \neq i,j,k}}^{\mathrm{N}} f_{i\nu}(\mathbf{d}^{ijk}) \right]^{-1} G_0(k - p_i) G_0(k - p_j) G_0(k - p_k) \tag{15}$$

valid for $G_0(k) = [ik_0 - (\mathbf{k}^2/2 - \mu)]^{-1}$ in two dimensions [12]. Note that $f_{i\nu}(\mathbf{d}^{ijk}) = f_{j\nu}(\mathbf{d}^{ijk}) = f_{k\nu}(\mathbf{d}^{ijk})$.

A remarkable explicit result has been obtained by Feldman et al. [1] in the static case ($q_{i0} = 0$ for all i). In that case they show that I_{N} vanishes identically for $\mathrm{N} > 2$ when all disks with radius k_F around the points $\mathbf{p}_1, \dots, \mathbf{p}_{\mathrm{N}}$ have at least one point in common! This implies that $\Pi_{\mathrm{N}}(q_1, \dots, q_{\mathrm{N}})$ vanishes identically for $q_{i0} = 0$ and small $\mathbf{q}_i$, if $\mathrm{N} > 2$, which is compatible with Eq. (6) since $D(\epsilon)$ is constant in two dimensions.

In the following we will derive more explicit formulae for 3-loops, and will analyze the behavior of Π_{N} and Π_{N}^S in various important limits. To this end, it is useful to determine the point $\mathbf{d}^{ijk}$ explicitly as a function of p_i, p_j, p_k. Writing the inhomogeneous linear system of equations for $\mathbf{d}^{ijk}$ in matrix form and inverting the matrix, one obtains

$$\mathbf{d}^{ijk} = \frac{1}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \left[\frac{1}{2} (\mathbf{p}_k^2 - \mathbf{p}_i^2) + i(p_{k0} - p_{i0}) \right] (\mathbf{p}_j - \mathbf{p}_i)^{\perp} \; + \; j \leftrightarrow k \tag{16}$$

Here $\mathbf{p}^{\perp} = (p_x, p_y)^{\perp} = (-p_y, p_x)$ denotes a rotation by an angle $\pi/2$ and $j \leftrightarrow k$ means exchange of indices j and k. Note that $|\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)|$ is twice the area of the triangle

with vertices $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$. If $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ are collinear, this area vanishes and the point $\mathbf{d}^{ijk}$ tends to infinity. Using (16), the coefficient $f_{i\nu}(\mathbf{d}^{ijk})$ appearing in the reduction formula (14) can be expressed explicitly as

$$\begin{aligned} f_{i\nu}(\mathbf{d}^{ijk}) &= \frac{1}{2}(\mathbf{p}_i^2 - \mathbf{p}_\nu^2) + i(p_{i0} - p_{\nu 0}) \\ &+ \left\{ \left[\frac{1}{2}(\mathbf{p}_k^2 - \mathbf{p}_i^2) + i(p_{k0} - p_{i0}) \right] \frac{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_\nu - \mathbf{p}_i)}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} + j \leftrightarrow k \right\} \end{aligned} \tag{17}$$

We conclude this section by defining a complex function of p_i, p_j, p_k that will be useful in the following evaluations:

$$\bar{z}_{ijk} = \bar{x}_{ijk} + i\bar{y}_{ijk} = \frac{\det(\mathbf{p}_j - \mathbf{d}^{ijk}, \mathbf{p}_i - \mathbf{d}^{ijk})}{|\mathbf{p}_j - \mathbf{p}_i|} = \det\left(\mathbf{d}^{ijk} - \mathbf{p}_i, \frac{\mathbf{p}_j - \mathbf{p}_i}{|\mathbf{p}_j - \mathbf{p}_i|}\right) \tag{18}$$

Inserting $\mathbf{d}^{ijk}$, one obtains

$$\begin{aligned} \bar{x}_{ijk} &= \frac{|\mathbf{p}_j - \mathbf{p}_i|}{2\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} (\mathbf{p}_j - \mathbf{p}_k) \cdot (\mathbf{p}_k - \mathbf{p}_i) \\ \bar{y}_{ijk} &= \frac{1}{\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_i)} \frac{\mathbf{p}_j - \mathbf{p}_i}{|\mathbf{p}_j - \mathbf{p}_i|} \cdot [(\mathbf{p}_k - \mathbf{p}_i)(p_{j0} - p_{k0}) - (\mathbf{p}_j - \mathbf{p}_k)(p_{k0} - p_{i0})] \end{aligned} \tag{19}$$

The modulus of $\bar{x}_{ijk}$ is twice the distance between the point $\mathrm{Re}\,\mathbf{d}^{ijk}$ and the straight line connecting $\mathbf{p}_i$ and $\mathbf{p}_j$. Note the useful identity

$$(\mathbf{d}^{ijk} - \mathbf{p}_i)^2 - \bar{z}_{ijk}^2 = \left[\frac{1}{2}|\mathbf{p}_j - \mathbf{p}_i| + i\,\frac{p_{j0} - p_{i0}}{|\mathbf{p}_j - \mathbf{p}_i|}\right]^2 \tag{20}$$

4. The 3-loop

In this section we derive an explicit formula for the 3-loop and analyze its behavior in various important limits.

We first derive an explicit expression for the functions $w_{ij}(s)$, $i, j \in \{1, 2, 3\}$, $i \neq j$. Equation (12) can also be written as

$$[|\mathbf{p}_j - \mathbf{p}_i| z + \bar{z}_{ij}]^2 + (\mathbf{d} - \mathbf{p}_i)^2 - \bar{z}_{ij}^2 = s^2 \tag{21}$$

where $\bar{z}_{ij} = \bar{x}_{ij} + \bar{y}_{ij} \equiv \bar{z}_{ijk}$ with $k \in \{1, 2, 3\}$, $k \neq i, j$. Making the ansatz

$$w_{ij}(s) = \frac{z_{ij}(s) - \bar{z}_{ij}}{|\mathbf{p}_j - \mathbf{p}_i|} \tag{22}$$

with a complex function $z_{ij}(s) = x_{ij}(s) + iy_{ij}(s)$, and using relation (20), one obtains the equation

$$z_{ij}^2(s) + \left[\frac{1}{2}|\mathbf{p}_j - \mathbf{p}_i| + i\,\frac{p_{j0} - p_{i0}}{|\mathbf{p}_j - \mathbf{p}_i|}\right]^2 = s^2 \tag{23}$$

Splitting real and imaginary parts, one obtains two real equations for $x_{ij}(s)$ and $y_{ij}(s)$,

$$\begin{aligned} x_{ij}^2(s) - y_{ij}^2(s) &= s^2 - \frac{1}{4}|\mathbf{p}_j - \mathbf{p}_i|^2 + \frac{(p_{j0} - p_{i0})^2}{|\mathbf{p}_j - \mathbf{p}_i|^2} \equiv a_{ij}(s) \\ x_{ij}(s)\, y_{ij}(s) &= -\frac{1}{2}(p_{j0} - p_{i0}) \end{aligned} \tag{24}$$

and the condition (13) implies

$$(p_{j0} - p_{i0})\, x_{ij}(s) - \frac{1}{2}|\mathbf{p}_j - \mathbf{p}_i|^2\, y_{ij}(s) > 0 \tag{25}$$

The two equations (24) define intersecting hyperbolas in the (x, y)-plane. Note that the second equation does not depend on the variable s. For any s there is exactly one intersection point satisfying the condition (25), given by

$$\begin{aligned} x_{ij}(s) &= \mathrm{sgn}(p_{j0} - p_{i0})\frac{1}{\sqrt{2}}\sqrt{\sqrt{[a_{ij}(s)]^2 + (p_{j0} - p_{i0})^2} + a_{ij}(s)} \\ y_{ij}(s) &= -\frac{1}{\sqrt{2}}\sqrt{\sqrt{[a_{ij}(s)]^2 + (p_{j0} - p_{i0})^2} - a_{ij}(s)} \end{aligned} \tag{26}$$

All the square-roots in (26) have real positive arguments.

It is now obvious that the curve $\gamma_{ij} = \{w_{ij}(s) | 0 \le s \le 1\}$ is connected and sweeps out an angle smaller than π with respect to the origin in the complex plane. Hence

$$\int_{\gamma_{ij}} \frac{dz}{z} = \ln\left[\frac{w_{ij}(1)}{w_{ij}(0)}\right] \tag{27}$$

where ln is the main-branch of the complex logarithm, which is real and continuous on the positive real axis and has a branch cut on the negative real axis. Hence,

$$I_3(p_1, p_2, p_3) = \frac{1}{2\pi i \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \sum_{\substack{i,j=1 \\ i\neq j}}^{3} s_{ij} \ln\left(\frac{z_{ij}(1) - \bar{z}_{ij}}{z_{ij}(0) - \bar{z}_{ij}}\right) \tag{28}$$

We have thus obtained an explicit formula for the 3-loop as a function of p_1, p_2, p_3, which is useful in particular for the evaluation of Feynman diagrams with 3-loops as subdiagrams. Note that the result $I_3 = 0$ in the static limit (for momenta such that unit disks around them have at least one point in common) is not obvious from Eq. (28), and is obtained only as a consequence of cancellations in the sum over i, j.

We now analyze the behavior of single 3-loops and the symmetrized 3-loop in the small-q limit defined in Sec. 2. With the choice $p_1 = 0$, the substitution $q_i \mapsto \lambda q_i$ implies $p_i \mapsto \lambda p_i$. The function $\bar{y}_{ij}$ is invariant under this substitution, while $\bar{x}_{ij}$ becomes a homogeneous function of λ of order one, i.e. $\bar{x}_{ij} \mapsto \lambda \bar{x}_{ij}$. For $\lambda \to 0$, the functions $x_{ij}(s)$ and $y_{ij}(s)$ reduce to

$$
\begin{aligned}
x_{ij}(s) \quad &\to \quad \mathrm{sgn}(p_{j0}-p_{i0})\sqrt{s^2 + (p_{j0}-p_{i0})^2/|\mathbf{p}_j-\mathbf{p}_i|^2} \\
y_{ij}(s) \quad &\to \quad -\frac{\lambda}{2}\,\frac{|\mathbf{p}_j-\mathbf{p}_i|}{\sqrt{1+s^2|\mathbf{p}_j-\mathbf{p}_i|^2/(p_{j0}-p_{i0})^2}}
\end{aligned}
\tag{29}
$$

Hence $x_{ij}(s)$ is of order one for $\lambda \to 0$ while $y_{ij}(s)$ is of order λ. To avoid confusion, we emphasize that we consider the small-q limit for *finite* fixed ratios between the various momenta and frequencies. The following small-q expansion is valid only if energy/momentum ratios $\frac{p_{j0}-p_{i0}}{|\mathbf{p}_j-\mathbf{p}_i|}$ etc. do not vanish. Hence, one cannot recover the static limit from the asymptotic expressions. The expansion does hold, however, in the dynamical limit, where energy/momentum ratios tend to infinity.

Using the power-series expansion $\ln(1-z) = -\sum_{n=1}^{\infty} \frac{1}{n} z^n$, we can expand

$$
\ln\left(\frac{w_{ij}(1)}{w_{ij}(0)}\right) = \ln\left[x_{ij}(s) - i\bar{y}_{ij}\right] - \sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{\bar{x}_{ij} - i y_{ij}(s)}{x_{ij}(s) - i\bar{y}_{ij}}\right]^n \Bigg|_{s=0}^{s=1}
\tag{30}
$$

The n-th term in the sum is of order λ^n. To compute I_3, we have to sum over all pairs (i,j) with $i,j \in \{1,2,3\}$ and $i \neq j$. Since

$$
\bar{x}_{ji} = -\bar{x}_{ij}\,,\ \bar{y}_{ji} = -\bar{y}_{ij} \quad \text{and} \quad x_{ji}(s) = -x_{ij}(s)\,,\ y_{ji}(s) = y_{ij}(s)\,,
\tag{31}
$$

the first term in (30) cancels when the contributions from γ_{ij} and γ_{ji} are subtracted (see Eq. (28)), i.e.

$$
\int_{\gamma_{ij}} \frac{dz}{z} - \int_{\gamma_{ji}} \frac{dz}{z} = \sum_{n=1}^{\infty} \frac{1}{n}\left\{\left[\frac{\bar{x}_{ij} + i y_{ij}(s)}{x_{ij}(s) - i\bar{y}_{ij}}\right]^n - \left[\frac{\bar{x}_{ij} - i y_{ij}(s)}{x_{ij}(s) - i\bar{y}_{ij}}\right]^n\right\} \Bigg|_{s=0}^{s=1}
\tag{32}
$$

The leading term for small λ is the one with $n = 1$. Neglecting the other terms, one obtains after some elementary algebra (see Appendix A),

$$
\begin{aligned}
I_3(p_1,p_2,p_3) \to \lambda^{-1} &\frac{1}{2\pi i\,\det(\mathbf{p}_2-\mathbf{p}_1,\mathbf{p}_3-\mathbf{p}_1)}\,\frac{1}{1+(\mathrm{Im}\,\mathbf{d})^2} \\
&\times \sum_{(i,j)=(1,2),(2,3),(3,1)} \frac{\det(\mathrm{Im}\,\mathbf{d},\mathbf{p}_j-\mathbf{p}_i)}{\sqrt{1+|\mathbf{p}_j-\mathbf{p}_i|^2/(p_{j0}-p_{i0})^2}}
\end{aligned}
\tag{33}
$$

with corrections of order one. Hence, a single 3-loop generally diverges as λ^{-1} for $\lambda \to 0$, just as expected from simple power-counting.

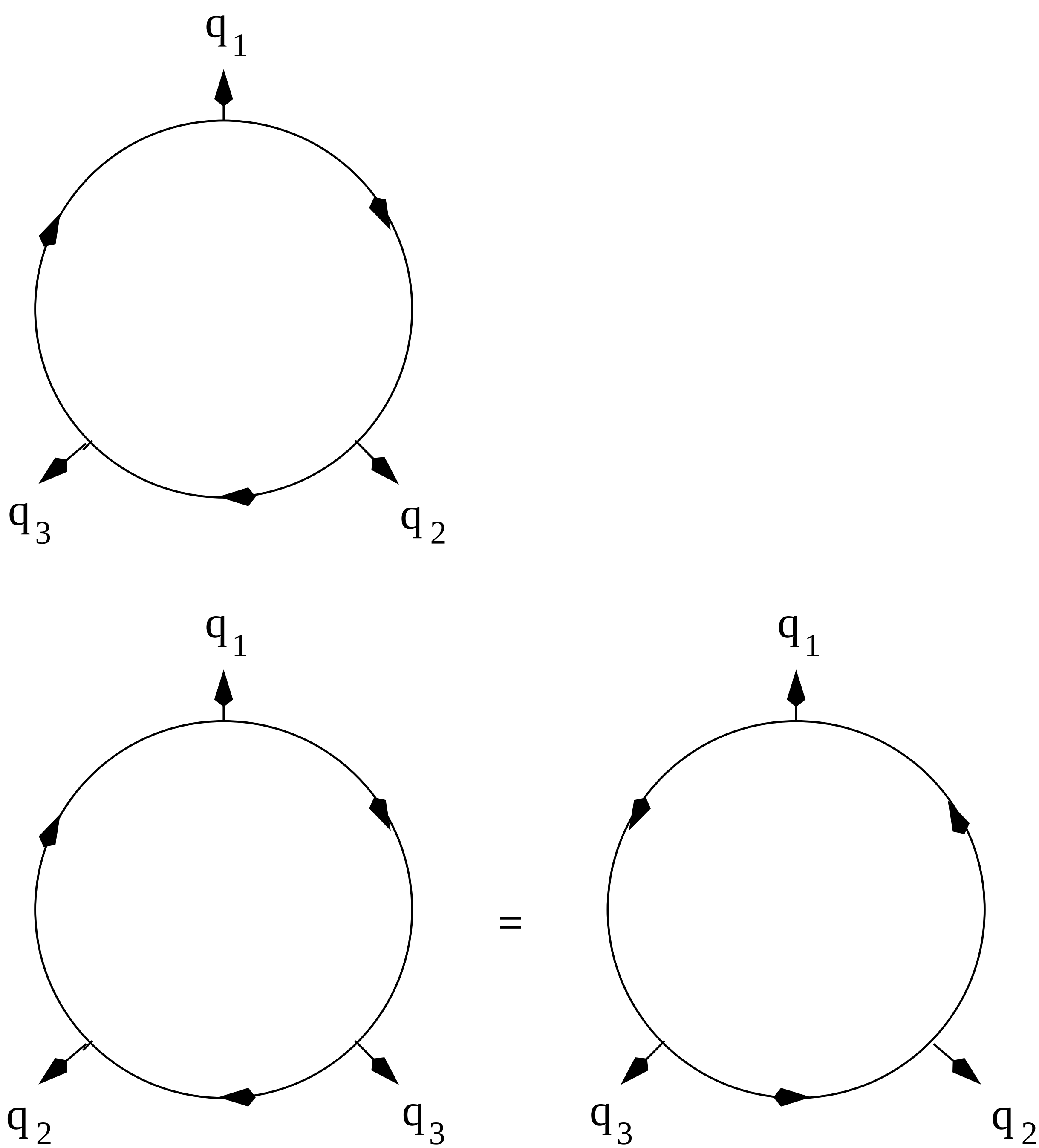

Figure 2: The two distinct permutations for the 3-loop.

We now determine the symmetrized 3-loop for small energies and momenta. There are only two non-equivalent permutations of q_1, q_2, q_3 (see Fig. 2), corresponding to a change of sign of p_1, p_2, p_3. The symmetrized loop Π_3^S can therefore be expressed as

$$\Pi_3^S(q_1, q_2, q_3) = \frac{1}{2}\left[I_3(p_1, p_2, p_3) + I_3(-p_1, -p_2, -p_3)\right] \equiv I_3^S(p_1, p_2, p_3) \tag{34}$$

It is easy to see that

$$\bar{x}_{ij} \mapsto \bar{x}_{ij}\;,\; \bar{y}_{ij} \mapsto -\bar{y}_{ij} \quad \text{and} \quad x_{ij}(s) \mapsto -x_{ij}(s)\;,\; y_{ij}(s) \mapsto y_{ij}(s) \quad \text{for} \quad p_i \mapsto -p_i. \tag{35}$$

Hence, the first order terms cancel when summing permuted loops, and the leading contributions come from $n = 2$ in the expansion (32), i.e.

$$\Pi_3^S(q_1, q_2, q_3) \to \frac{1}{4\pi i \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \sum_{(i,j)=(1,2),(2,3),(3,1)} \left[\left(X_{ij}^+(s)\right)^2 - \left(X_{ij}^-(s)\right)^2\right]\Bigg|_{s=0}^{s=1} \tag{36}$$

where

$$X_{ij}^{\pm}(s) \equiv \frac{\bar{x}_{ij} \pm i y_{ij}(s)}{x_{ij}(s) - i\bar{y}_{ij}}\,. \tag{37}$$

After some simple algebra (see Appendix A) one obtains

$$\Pi_3^S(q_1, q_2, q_3) \to \frac{1}{\pi \det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)} \frac{1}{[s^2 + (\operatorname{Im}\mathbf{d})^2]^2} \sum_{(i,j)=(1,2),(2,3),(3,1)} \bar{x}_{ij}$$
$$\times \left[[s^2 + (\operatorname{Im}\mathbf{d})^2] \frac{|\mathbf{p}_j - \mathbf{p}_i|/2}{\sqrt{1 + s^2|\mathbf{p}_j - \mathbf{p}_i|^2/(p_{j0} - p_{i0})^2}} - |p_{j0} - p_{i0}| \sqrt{s^2 + \frac{(p_{j0} - p_{i0})^2}{|\mathbf{p}_j - \mathbf{p}_i|^2}} \right]\Bigg|_{s=0}^{s=1} \tag{38}$$

with corrections of order λ. The symmetrized loop is thus *finite* (of order one) for $\lambda \to 0$, i.e. the divergence present in single loops is cancelled when summing permutations. Note also that $\Pi_3^S(q_1, q_2, q_3)$ is obviously *real* in the small-q limit.

Next we analyze the behavior of the symmetrized 3-loop for *one* vanishing external momentum variable, say $\mathbf{q}_1$, while the other momenta and all energy variables remain finite. Noting that $\det(\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1)$ is of order $\mathbf{q}_1$ in that limit, it is easy to see that $X_{ij}^{\pm}(s) = O(|\mathbf{q}_1|)$ for all i, j. Hence, we can use the expansion (32) once again. For the symmetrized loop the first order terms cancel, and the leading contribution is given by (36), which is of order $|\mathbf{q}_1|$.

We finally consider the dynamical limit, where all momenta scale to zero (with a scaling factor λ) at fixed finite energy variables. In that limit $X_{ij}^{\pm}(s)$ is of order λ^2, so that the leading contribution to the symmetrized loop is again given by (36). Solving for $z_{ij}(s)$ for small λ, one can easily show that the difference $X_{ij}^{\pm}(1) - X_{ij}^{\pm}(0)$ is only of order

λ^4. Since $\det(\mathbf{p}_2-\mathbf{p}_1, \mathbf{p}_3-\mathbf{p}_1)$ is obviously of order λ^2 in the dynamical limit, one obtains the result

$$\Pi_3^S((q_{10}, \lambda\mathbf{q}_1), (q_{20}, \lambda\mathbf{q}_2), (q_{30}, \lambda\mathbf{q}_3)) = O(\lambda^4) \quad \text{for} \quad \lambda \to 0 \tag{39}$$

5. Permutations of external energy-momentum variables

Consider a term in the reduction formula (14) with fixed values i, j, k. Let $\{\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_M\} \subset \{q_1, \ldots, q_N\}$ be the subset of those external energy-momentum variables that are attached to the loop between the lines labelled by p_j and p_k, as shown in Fig. 3 (the case of variables attached between p_i and p_j or p_k and p_i is completely analogous). Define $\tilde{p}_0 = p_j$, $\tilde{p}_1 = p_{j+1}$, $\tilde{p}_2 = p_{j+2}, \ldots, \tilde{p}_M = p_{k-1}$, $\tilde{p}_{M+1} = p_k$.

Our aim is to show that the sum

$$\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_M) \equiv \sum_{\tilde{P}} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{\tilde{P}}} \tag{40}$$

over all permutations $\tilde{P}$ of $\tilde{q}_0, \ldots, \tilde{q}_M$ is of order one in the small-q limit, and vanishes with a power $2M$ in the dynamical limit. Here

$$f_\mu^{\tilde{P}} \equiv f_{i,j+\mu}^{\tilde{P}}(\mathbf{d}^{ijk}) = (\tilde{\mathbf{p}}_\mu^{\tilde{P}} - \mathbf{p}_i) \cdot \mathbf{d}^{ijk} + \frac{1}{2}[\mathbf{p}_i^2 - (\tilde{\mathbf{p}}_\mu^{\tilde{P}})^2] + i(p_{i0} - \tilde{p}_{\mu 0}^{\tilde{P}}) \tag{41}$$

is constructed with the energy-momentum variable $\tilde{p}_\mu^{\tilde{P}}$ on the μ-th line resulting from the permutation $\tilde{P}$. Note that for a fixed permutation $\tilde{P}$, the product $\prod_{\mu=1}^{M}(1/f_\mu^{\tilde{P}})$ diverges as λ^{-M} for $\lambda \to 0$. Only the sum over all permutations turns out to be finite in that limit, due to systematic cancellations.

As a first step, we sum over permutations leading to sequences

$$\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_\alpha, \tilde{q}_0, \tilde{q}_{\alpha+1}, \ldots, \tilde{q}_M, \tag{42}$$

where $\tilde{q}_1, \ldots, \tilde{q}_M$ follow the order of their indices, while $\tilde{q}_0$ is placed between $\tilde{q}_\alpha$ and $\tilde{q}_{\alpha+1}$. We denote these particular permutations by (α), where $\alpha \in \{0, 1, \ldots, M\}$. Obviously

$$\tilde{p}_\mu^{(\alpha)} = \begin{cases} \tilde{p}'_\mu = \tilde{p}_{\mu+1} - \tilde{q}_0 & \text{for} \quad \mu \le \alpha \\ \tilde{p}_\mu & \text{for} \quad \mu > \alpha \end{cases} \tag{43}$$

Define

$$\begin{aligned} f_\mu^{(\alpha)} \equiv f_{i,j+\mu}^{(\alpha)}(\mathbf{d}^{ijk}) &= (\tilde{\mathbf{p}}_\mu^{(\alpha)} - \mathbf{p}_i) \cdot \mathbf{d}^{ijk} + \frac{1}{2}[\mathbf{p}_i^2 - (\tilde{\mathbf{p}}_\mu^{(\alpha)})^2] + i(p_{i0} - \tilde{p}_{\mu 0}^{(\alpha)}) \\ &\equiv \begin{cases} f'_\mu & \text{for} \quad \mu \le \alpha \\ f_\mu & \text{for} \quad \mu > \alpha \end{cases} \end{aligned} \tag{44}$$

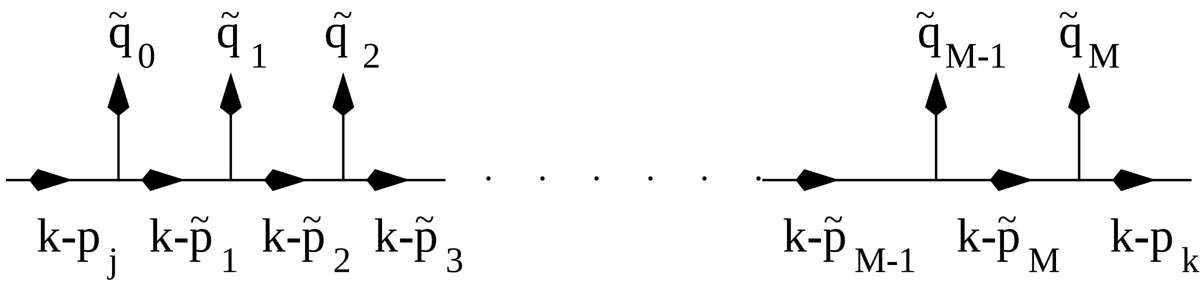

Figure 3: Part of an N-loop from p_j to p_k.

i.e. f_μ is constructed with $\tilde{p}_\mu$ and f'_μ with $\tilde{p}'_\mu = \tilde{p}_\mu + \tilde{q}_\mu - \tilde{q}_0$. For the sum over all permutations (α) we have derived the following

Lemma 1:

$$\sum_{\alpha=0}^{M} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} = \frac{1}{f_1} \sum_{\alpha=1}^{M} \tilde{\mathbf{q}}_0 \cdot (\tilde{\mathbf{q}}_1 + \ldots + \tilde{\mathbf{q}}_\alpha) \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} \tag{45}$$

Proof: Using $\tilde{p}'_\mu = \tilde{p}_{\mu+1} - \tilde{p}_1 + p_j$ and the identity $f_0 \equiv f_{ij}(\mathbf{d}^{ijk}) = 0$, it is easy to derive the relation

$$f_1 + f'_\mu = f_{\mu+1} + g_{\mu+1} \tag{46}$$

where

$$g_{\alpha+1} \equiv \tilde{\mathbf{q}}_0 \cdot (\tilde{\mathbf{p}}_{\alpha+1} - \tilde{\mathbf{p}}_1) = \tilde{\mathbf{q}}_0 \cdot (\tilde{\mathbf{q}}_1 + \ldots + \tilde{\mathbf{q}}_\alpha) \tag{47}$$

This yields

$$\left[\frac{1}{f_1} + \frac{1}{f'_\alpha}\right] \prod_{\substack{\mu=1 \\ \mu\neq\alpha}}^{M} \frac{1}{f_\mu^{(\alpha)}} = \frac{f_{\alpha+1} + g_{\alpha+1}}{f_1} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} \tag{48}$$

We now prove the identity (45) by starting from the right hand side. Relation (48) yields

$$\sum_{\alpha=1}^{M} \frac{g_{\alpha+1}}{f_1} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} = \sum_{\alpha=1}^{M} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} + \frac{1}{f_1} \sum_{\alpha=1}^{M} (f'_\alpha - f_{\alpha+1}) \prod_{\mu=1}^{M} \frac{1}{f_\mu^{(\alpha)}} \tag{49}$$

The second term reduces to $\prod_{\mu=1}^{M}(1/f_\mu)$, due to a cancellation of all terms but the first one in the α-sum (note that $f_{M+1} \equiv f_{ik}(\mathbf{d}^{ijk}) = 0$). This completes the proof of Lemma 1. For energy-momentum variables attached between p_i and p_j or between p_k and p_i, it can be proved in the same way.

For the following formal manipulations it will be helpful to represent products of inverse f-factors $\prod_{\mu=1}^{M}(1/f_\mu^{\tilde{P}})$ by the ordered sequence of momentum labels corresponding to the permutation $\tilde{P}$, i.e.

$$\prod_{\mu=1}^{M} \frac{1}{f_\mu^{\tilde{P}}} \equiv (\tilde{P}0, \tilde{P}1, \tilde{P}2, \ldots, \tilde{P}M) \tag{50}$$

With this notation, Lemma 1 can also be written as

$$\sum_{\alpha=0}^{M}(1,2,\ldots,\alpha,0,\alpha+1,\ldots,M) = \frac{1}{f_1}\sum_{\substack{\alpha,\rho=1\\ \rho\leq\alpha}}^{M}(\tilde{\mathbf{q}}_0\cdot\tilde{\mathbf{q}}_\rho)\,(1,2,\ldots,\alpha,0,\alpha+1,\ldots,M) \tag{51}$$

In particular, for $M=1$ one obtains

$$(0,1)+(1,0) = \frac{\tilde{\mathbf{q}}_0\cdot\tilde{\mathbf{q}}_1}{f_1}\,(1,0) \tag{52}$$

and for $M=2$,

$$(0,1,2)+(1,0,2)+(1,2,0) = \frac{1}{f_1}\left\{(\tilde{\mathbf{q}}_0\cdot\tilde{\mathbf{q}}_1)\,(1,0,2)+[\tilde{\mathbf{q}}_0\cdot(\tilde{\mathbf{q}}_1+\tilde{\mathbf{q}}_2)]\,(1,2,0)\right\} \tag{53}$$

The sum over permutations (α) has reduced the degree of divergence for $\lambda\to 0$ by one power. We now consider permutations (α,β) with $\beta\leq\alpha$ leading to sequences

$$\tilde{q}_2,\tilde{q}_3,\ldots,\tilde{q}_\beta,\tilde{q}_1,\tilde{q}_{\beta+1},\ldots,\tilde{q}_\alpha,\tilde{q}_0,\tilde{q}_{\alpha+1},\ldots,\tilde{q}_M \tag{54}$$

i.e. $\tilde{q}_2,\ldots,\tilde{q}_M$ follow the order of their indices while $\tilde{q}_1$ is placed at an arbitrary position before $\tilde{q}_0$ in the sequence. The corresponding energy-momentum variables on fermion lines become

$$\tilde{p}_\mu^{(\alpha,\beta)} = \begin{cases} \tilde{p}''_\mu = \tilde{p}_{\mu+2}-\tilde{q}_0-\tilde{q}_1 & \text{for} \quad \mu\leq\beta \\ \tilde{p}'_\mu = \tilde{p}_{\mu+1}-\tilde{q}_0 & \text{for} \quad \beta<\mu\leq\alpha \\ \tilde{p}_\mu & \text{for} \quad \mu>\alpha \end{cases} \tag{55}$$

and the associated product of inverse f-factors is

$$\prod_{\mu=1}^{M}\frac{1}{f_\mu^{(\alpha,\beta)}} = (2,3,\ldots,\beta,1,\beta+1,\ldots,\alpha,0,\alpha+1,\ldots,M) \tag{56}$$

The sum over all permutations (α,β) with fixed α and $\beta\leq\alpha$, can be rewritten using

Lemma 2:

$$\sum_{\beta=0}^{\alpha-1}\prod_{\mu=1}^{M}\frac{1}{f_\mu^{(\alpha,\beta)}} = \frac{1}{f'_1}\sum_{\beta=1}^{\alpha-1}\tilde{\mathbf{q}}_1\cdot(\tilde{\mathbf{q}}_2+\ldots+\tilde{\mathbf{q}}_{\beta+1})\prod_{\mu=1}^{M}\frac{1}{f_\mu^{(\alpha,\beta)}} + \frac{1}{f'_1}\prod_{\mu=1}^{\alpha-1}\frac{1}{f''_\mu}\prod_{\mu=\alpha+1}^{M}\frac{1}{f_\mu} \tag{57}$$

The second term on the right hand side can be written symbolically as

$$\frac{1}{f'_1}\prod_{\mu=1}^{\alpha-1}\frac{1}{f''_\mu}\prod_{\mu=\alpha+1}^{M}\frac{1}{f_\mu} = \frac{1}{f'_1}(2,3,\ldots,\alpha,[0,1],\alpha+1,\ldots,M) \tag{58}$$

where $[0,1]$ stands for the sum of two energy-momentum variables, $\tilde{q}_0+\tilde{q}_1$, i.e. the product of inverse f-factors above corresponds to the sequence $\tilde{q}_2,\tilde{q}_3,\ldots,\tilde{q}_\alpha,\tilde{q}_0+\tilde{q}_1,\tilde{q}_{\alpha+1},\ldots,\tilde{q}_M$.

Proof: The proof of Lemma 2 is almost identical to the one for Lemma 1. A difference arises only in the last step, where now the "boundary term" (58) contributes, because $f_\alpha \neq 0$ replaces the factor $f_{M+1} = 0$ (see Eq. (49) and below).

For example, for $M = 2$, $\alpha = 1$, Lemma 2 yields the trivial identity

$$(1,0,2) = \frac{1}{f_1'}\,([0,1],2) \tag{59}$$

and for $M = 2$, $\alpha = 2$,

$$(1,2,0) + (2,1,0) = \frac{1}{f_1'}\,(\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2)\,(2,1,0) + \frac{1}{f_1'}\,(2,[0,1]) \tag{60}$$

An example for $M = 3$ can be found in Appendix B.

Lemma 1 and Lemma 2 have been formulated and derived for $\tilde{q}_0$ and $\tilde{q}_1$ "running" from left to right, with a specific fixed order of the other variables. Analogous identities hold of course for any permutation of running and fixed variables, e.g. for $\tilde{q}_3$ running with some fixed order of all other variables. In general, if $\tilde{q}_\rho$ is the running variable, the explicit prefactor $1/f_1$ in Lemma 1 and $1/f_1'$ in Lemma 2 has to be replaced by a factor $1/h_\rho$, where

$$h_\rho = (\mathbf{p}_j + \tilde{\mathbf{q}}_\rho - \mathbf{p}_i)\cdot \mathbf{d}^{ijk} + \frac{1}{2}[\mathbf{p}_i^2 - (\mathbf{p}_j + \tilde{\mathbf{q}}_\rho)^2] + i[p_{i0} - (p_{j0} + \tilde{q}_{\rho 0})] \tag{61}$$

i.e. an f-function constructed with p_i and $p_j + \tilde{q}_\rho$. Note that $f_1 = h_0$ and $f_1' = h_1$ by definition.

We are now ready to prove the main result of this section:

Theorem 1: *The sum* $\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_M) = \sum_{\tilde{P}} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{\tilde{P}}}$ *can be written as a sum over fractions with numerators*

$$(\tilde{\mathbf{q}}_{\sigma_1} \cdot \tilde{\mathbf{q}}_{\sigma_1'})\,(\tilde{\mathbf{q}}_{\sigma_2} \cdot \tilde{\mathbf{q}}_{\sigma_2'})\;\ldots\;(\tilde{\mathbf{q}}_{\sigma_M} \cdot \tilde{\mathbf{q}}_{\sigma_M'}) \tag{62}$$

where $\sigma_i \neq \sigma_i'$, *and denominators*

$$h_0\,h_1\,\ldots\,h_M \times \prod_{1}^{M-1} f\text{-factors} \tag{63}$$

where the f*-factors are constructed with* p_i *and* p_j + *some variables* $\tilde{q}$. *In each numerator each momentum variable from* $\{\tilde{\mathbf{q}}_0, \ldots, \tilde{\mathbf{q}}_M\}$ *appears at least once.*

Proof: The following *algorithm* reduces the sum $\sum_{\tilde{P}} \prod_{\mu=1}^{M} \frac{1}{f_\mu^{\tilde{P}}}$ to the form described in the theorem. We proceed by induction with respect to the number of energy-momentum variables $M+1$. We will prove that the algorithm works for the case $M = 1$ directly, and

for general M under the assumption that it works for less than $M+1$ variables.

Step 1: Write down all permutations of $(0,1,\ldots,M)$, and apply Lemma 1 to groups with a fixed order of $1,\ldots,M$, and 0 running over all possible positions. This yields

$$\sum_{\tilde{P}} \prod_{\mu=1}^{M} \frac{1}{f_{\mu}^{\tilde{P}}} = \sum_{\rho=1}^{M} \frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_\rho}{h_0} \sum (\ldots,\rho,\ldots,0,\ldots) \tag{64}$$

where, for fixed ρ, all products $(\ldots,\rho,\ldots,0,\ldots)$ with ρ to the left of 0 contribute exactly once. For $M=1$ one gets simply

$$\tilde{S}_1(\tilde{q}_0,\tilde{q}_1) = (0,1)+(1,0) = \frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0}\,(1,0) = \frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0\, h_1} \tag{65}$$

i.e. for this case Theorem 1 is already proven.

Step 2: The coefficients $\sum(\ldots,\rho,\ldots,0,\ldots)$ of $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_\rho}{h_0}$ can be further reduced by applying Lemma 2 to groups with fixed positions of all variables except ρ and ρ running from the first place to the place before and nearest to 0. This yields a sum

$$\sum_{\substack{\rho'=1\\ \rho'\neq 0,\rho}}^{M} \frac{\tilde{\mathbf{q}}_\rho \cdot \tilde{\mathbf{q}}_{\rho'}}{h_\rho} \sum (\ldots,\rho',\ldots,\rho,\ldots,0,\ldots) + \frac{1}{h_\rho} \sum (\ldots,[0,\rho],\ldots) \tag{66}$$

where the coefficient $\sum(\ldots,\rho',\ldots,\rho,\ldots,0,\ldots)$ is a sum over all permutations with ρ' to the left of ρ and ρ to the left of 0. The "boundary term" $\sum(\ldots,[0,\rho],\ldots)$ is a sum over all permutations of products of $M-1$ f-factors with M energy-momentum variables, i.e. all variables $\tilde{q}_\mu$ with $\mu \neq 0,\rho$ and $\tilde{q}_0+\tilde{q}_\rho$. According to our induction hypothesis we can apply Theorem 1 to this term, which, with the prefactor $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_\rho}{h_0 h_\rho}$ obtained in Step 1 and Step 2, leads to a sum over fractions with the required structure.

Step 3: To reduce the first sum of terms obtained in Step 2, we apply Lemma 2 to the coefficients $\sum(\ldots,\rho',\ldots,\rho,\ldots,0,\ldots)$, this time with ρ' running from the first place to ρ. This yields sums

$$\sum_{\substack{\rho''=1\\ \rho''\neq 0,\rho,\rho'}}^{M} \frac{\tilde{\mathbf{q}}_{\rho'} \cdot \tilde{\mathbf{q}}_{\rho''}}{h_{\rho'}} \sum (\ldots,\rho'',\ldots,\rho',\ldots,\rho,\ldots,0,\ldots) + \frac{1}{h_{\rho'}} \sum (\ldots,[\rho,\rho'],\ldots,0,\ldots) \tag{67}$$

Again, the boundary term leads back to a case with one variable less, which is reducible to the desired form by virtue of the induction hypothesis.

Steps 4-M: The first sum of terms from Step 3 can be reduced by another application of Lemma 2, and so on. The boundary terms always lead back to a case with a reduced number of energy-momentum variables. The algorithm terminates after the M-th step,

where coefficients $(\rho^{(M-1)}, \rho^{(M-2)}, \dots, \rho'', \rho', \rho, 0)$ with a fixed order appear, where $\rho^{(M-1)}$ is the only variable that has not "run" (in the sense of Lemma 2). These coefficients correspond to inverse products of M f-factors, the first of which is $h_{\rho^{(M-1)}}$. It is clear that each momentum variable appears at least once in each product of M scalar products generated by the algorithm. This completes the proof of the theorem.

For $M = 2$, for example, the algorithm yields

$$\tilde{S}_2(\tilde{q}_0, \tilde{q}_1, \tilde{q}_2) \;=\; (\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1)\,(\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2)\,\frac{1}{h_0\,h_1\,h_2}\left[\frac{1}{h_{0,1}} + \frac{1}{h_{1,2}}\right] + \text{cyclic permutations} \tag{68}$$

where $h_{\rho,\rho'}$ is an f-function constructed with p_i and $p_j + \tilde{q}_\rho + \tilde{q}_{\rho'}$. In Appendix B the algorithm is presented at work for the case $M = 3$.

The following important corollaries follow directly from Theorem 1.

Corollary 1:
$\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_M)$ is *finite* (of order one) and *real* in the small-q limit, where all momentum and energy variables scale to zero as λ.

Corollary 2:
$\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_M)$ is of order $\tilde{\mathbf{q}}_\rho$ for $\tilde{\mathbf{q}}_\rho \to 0$ if $\tilde{q}_{\rho 0}$ and all $\tilde{q}_\mu$ with $\mu \neq \rho$ remain finite.

Corollary 3:
In the dynamical limit, where all momenta scale to zero as λ at fixed finite energy variables, the sum $\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_M)$ vanishes as λ^{2M}.

Proof: Theorem 1 tells us that $\tilde{S}_M(\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_M)$ is a sum of fractions with M scalar products of momenta in the numerator and $2M$ f-factors in the denominator. The f-factors vanish linearly in the small-q limit (where momenta and energies tend both to zero), and are purely imaginary to leading order in λ. This yields Corollary 1. The f-factors remain finite if momenta vanish at finite energies. This yields the Corollaries 2 and 3, where for the former the observation that in each numerator each momentum variable appears at least once is crucial.

Actually Corrollary 2 holds already for a sum over all positions of $\tilde{q}_\rho$ with fixed order of the other variables (instead of summing all permutations), and follows directly from Lemma 1.

In the following sections we will use these results to control the behavior of symmetrized N-loops in the above limits.

6. The N-loop

Any N-loop can be computed by inserting the explicit expression for the 3-loop given in Sec. 4 into the reduction formula (14), with $f_{i\nu}(\mathbf{d}^{ijk})$ from (17).

In the small-q limit, where all momenta and energy variables scale to zero as λ, the unsymmetrized N-loop diverges as $\lambda^{2-\mathrm{N}}$, since I_3 diverges as λ^{-1} and the product of N$-$3 factors $f_{i\nu}(\mathbf{d}^{ijk})$ vanishes as $\lambda^{\mathrm{N}-3}$. We will now show that the symmetrized N-loop (7) remains finite in the small-q limit.

Symmetrizing the reduction formula (14) for the N-loop, one can write

$$\Pi^S_{\mathrm{N}}(q_1,\dots,q_{\mathrm{N}}) \;=\; \mathcal{S} \sum_{1\leq i<j<k\leq N} S_{ij}\, S_{jk}\, S_{ki}\, I_3(p_i,p_j,p_k) \tag{69}$$

Here S_{ij}, S_{jk} and S_{ki} are symmetrized products of inverse f-factors, e.g.

$$S_{jk} \equiv \begin{cases} \frac{1}{(M+1)!}\,\tilde{S}_M & \text{for} \quad M = k-j-1 \geq 1 \\ 1 & \text{for} \quad k = j+1 \end{cases} \tag{70}$$

with $\tilde{S}_M$ as defined in Sec. 5. Note that symmetrizing once or twice, or first symmetrizing partially (with respect to a subset of variables) and then completely (by applying $\mathcal{S}$), or vice versa, always yields the same result. In particular,

$$\Pi^S_{\mathrm{N}}(q_1,\dots,q_{\mathrm{N}}) = \frac{1}{2}\left[\Pi^S_{\mathrm{N}}(q_1,\dots,q_{\mathrm{N}}) + (p_1,\dots,p_{\mathrm{N}}) \mapsto (-p_1,\dots,-p_{\mathrm{N}})\right] \tag{71}$$

Since $f_{i\nu}(\mathbf{d}^{ijk}) \mapsto f^*_{i\nu}(\mathbf{d}^{ijk})$ and thus $\tilde{S}_M \mapsto \tilde{S}^*_M$ for $(p_1,\dots,p_{\mathrm{N}}) \mapsto (-p_1,\dots,-p_{\mathrm{N}})$, one can also write

$$\Pi^S_{\mathrm{N}}(q_1,\dots,q_{\mathrm{N}}) \;=\; \mathcal{S} \sum_{1\leq i<j<k\leq N} \Big[\,\mathrm{Re}(S_{ij}S_{jk}S_{ki})\, I^S_3(p_i,p_j,p_k) + i\,\mathrm{Im}(S_{ij}S_{jk}S_{ki})\, I^A_3(p_i,p_j,p_k)\Big] \tag{72}$$

where $I^A_3(p_i,p_j,p_k) \equiv \frac{1}{2}\, I_3(p_i,p_j,p_k) - I_3(-p_i,-p_j,-p_k)$. Note that unsymmetrized loops are complex for $\mathrm{N} > 2$, while symmetrized loops are *real*, since $I_{\mathrm{N}}(-p_1,\dots,-p_{\mathrm{N}}) = I^*_{\mathrm{N}}(p_1,\dots,p_{\mathrm{N}})$.

In the small-q limit, $\mathrm{Re}(S_{ij}\, S_{jk}\, S_{ki})$ is of order one, while $\mathrm{Im}(S_{ij}\, S_{jk}\, S_{ki})$ is of order λ (see Theorem 1 and Corollary 1 in Sec. 5). In Sec. 4 we have shown that I^S_3 is of order one and real in that limit, while I^A_3 diverges as λ^{-1} with a purely imaginary coefficient (see Eq. (33)). Hence $\Pi^S_{\mathrm{N}}(q_1,\dots,q_{\mathrm{N}})$ is *finite* in the small-q limit, i.e.

$$\Pi^S_{\mathrm{N}}(\lambda q_1,\dots,\lambda q_{\mathrm{N}}) = O(1) \quad \text{for} \quad \lambda \to 0 \tag{73}$$

We now analyse the dynamical limit, where all momenta scale to zero as λ at finite energy variables. In that limit $\mathrm{Re} f_{i\nu}(\mathbf{d}^{ijk})$ is of order λ^2, while $\mathrm{Im} f_{i\nu}(\mathbf{d}^{ijk})$ is of order

one (see, for example, Eq. (17). Theorem 1 then implies that $\mathrm{Re}(S_{ij}S_{jk}S_{ki})$ vanishes as $\lambda^{2(\mathrm{N}-3)}$, and $\mathrm{Im}(S_{ij}S_{jk}S_{ki})$ as $\lambda^{2(\mathrm{N}-2)}$. The symmetrized 3-loop has been shown to vanish as λ^4 in the dynamical limit in Sec. 4. Since $X^{\pm}_{ij}(s)$ is of order λ^2 and $X^{\pm}_{ij}(1) - X^{\pm}_{ij}(0)$ of order λ^4 (see Sec. 4), it is obvious from (32) that I^A_3 vanishes as λ^2 in the dynamical limit. Inserting these asymptotic power-laws in (72), one finds

$$\Pi^S_{\mathrm{N}}((q_{10}, \lambda\mathbf{q}_1), \ldots, (q_{\mathrm{N}0}, \lambda\mathbf{q}_{\mathrm{N}}) = O(\lambda^{2\mathrm{N}-2}) \quad \text{for} \quad \lambda \to 0 \tag{74}$$

We finally discuss the behavior of loops for a single vanishing momentum, say $\mathbf{q}_1$, while energy variables and other momenta remain finite. Note that this limit cannot be taken for 2-loops, since momentum conservation imposes $\mathbf{q}_2 = -\mathbf{q}_1$ in that case, i.e. $\mathbf{q}_2$ cannot remain finite if $\mathbf{q}_1$ tends to zero. Unsymmetrized loops with $\mathrm{N} \geq 3$ remain finite in the small $\mathbf{q}_1$-limit, since f-factors and 3-loops remain finite if the other variables remain finite. We have already seen in Sec. 4 that the symmetrized 3-loop vanishes linearly for $\mathbf{q}_1 \to 0$. This behavior holds also for the symmetrized N-loop, i.e.

$$\Pi^S_{\mathrm{N}}(q_1, \ldots, q_{\mathrm{N}}) = O(|\mathbf{q}_1|) \quad \text{for} \quad \mathbf{q}_1 \to 0 \tag{75}$$

This latter result can be established quite easily in any dimension by using the techniques reviewed in Refs. [2, 4], for example via Ward identities [9, 10].

The above results for the symmetrized loops have been checked numerically for $\mathrm{N} = 3, 4, \ldots, 7$. No cancellations beyond the order established here have been observed.

7. Conclusion

In summary, we have derived explicit formulae for fermion loops in two dimensions from an exact expression obtained recently by Feldman et al. [1]. The 3-loop is an elementary function of momenta and frequencies, and the N-loop can be expressed as a linear combination of 3-loops with coefficients that are rational functions of momentum and frequency variables. These formulae are very useful for the evaluation of Feynman diagrams for two-dimensional interacting Fermi systems where such loops appear as subdiagrams.

We have also analyzed to what extent divergencies in the low energy and small momentum limit cancel in the symmetrized loop, defined as a sum over permutations of momentum and frequency variables. A single loop was seen to diverge with a power $\mathrm{N}-2$ as expected from a simple scaling analysis. We have proved a reduction of the degree of divergence in the symmetrized N-loop by $\mathrm{N}-2$ powers, such that symmetrized loops and thus all N-point density correlation functions of the two-dimensional Fermi gas are generally finite in the low energy and small momentum limit. This drastic cancellation justifies

the frequent neglect of loops with more than two insertions assumed in Ward identity or bosonization methods [2, 4] for systems where interactions with small momentum transfers dominate the low-energy physics. In the dynamical limit, where momenta scale to zero at finite energy variables, we have shown that the symmetrized N-loop vanishes as the (2N−2)-th power of the scale parameter.

Acknowledgements:
We are grateful to C. Halboth, H. Knörrer and E. Trubowitz for valuable discussions.

Appendix A: 3-loop for small energies and momenta

In this appendix we present the algebra leading to the asymptotic results for the 3-loop in Sec. 4.

The first order term in (32) is

$$\frac{2i\, y_{ij}(s)}{x_{ij}(s) - i\bar{y}_{ij}}\bigg|_{s=0}^{s=1} = 2i\, \frac{[x_{ij}(s) + i\bar{y}_{ij}]\, y_{ij}(s)}{x_{ij}^2(s) + \bar{y}_{ij}^2}\bigg|_{s=0}^{s=1} \tag{76}$$

The denominator can be simplified to

$$x_{ij}^2(s) + \bar{y}_{ij}^2 = x_{ij}^2(s) + (\mathrm{Im}\,\mathbf{d})^2 - \left(\mathrm{Im}\,\mathbf{d} \cdot \frac{\mathbf{p}_j - \mathbf{p}_i}{|\mathbf{p}_j - \mathbf{p}_i|}\right)^2 = s^2 + (\mathrm{Im}\,\mathbf{d})^2 \tag{77}$$

where in the second step we have used the fact that $\mathbf{d}$ is a zero of the function $f_{ij}(\mathbf{k})$ defined in (10). Relation (24) yields the identity

$$\sum_{(i,j)=(1,2),(2,3),(3,1)} x_{ij}(s)\, y_{ij}(s) = 0 \tag{78}$$

and $\bar{y}_{ij}\, y_{ij}(0) = \frac{1}{2}\det(\mathrm{Im}\,\mathbf{d}, \mathbf{p}_j - \mathbf{p}_i)$ yields

$$\sum_{(i,j)=(1,2),(2,3),(3,1)} \bar{y}_{ij}\, y_{ij}(0) = 0 \tag{79}$$

Inserting the above relations in the first order term in (32) one immediately obtains the result (33).

The second order term in (32) is

$$\frac{2i\, \bar{x}_{ij}\, y_{ij}(s)}{[x_{ij}(s) - i\bar{y}_{ij}]^2}\bigg|_{s=0}^{s=1} = 2i\, \frac{\bar{x}_{ij}\, y_{ij}(s)[x_{ij}^2(s) - \bar{y}_{ij}^2 + 2i x_{ij}(s)\, \bar{y}_{ij}]}{[s^2 + (\mathrm{Im}\,\mathbf{d})^2]^2}\bigg|_{s=0}^{s=1} \tag{80}$$

where the last step follows from (77). Using Eq. (24), i.e. $x_{ij}(s)\, y_{ij}(s) = -\frac{1}{2}(p_{j0} - p_{i0})$, and the identity

$$\sum_{(i,j)=(1,2),(2,3),(3,1)} \bar{x}_{ij}\, \bar{y}_{ij}\, (p_{j0} - p_{i0}) = 0 \tag{81}$$

one obtains the result (38).

Appendix B: Permutations for $M = 3$

In this Appendix we carry out the permutation algorithm constructed in Sec. 5 for the case $M = 3$.

Step 1: Write down all permutations of $(0,1,2,3)$, i.e. 24 terms, and apply Lemma 1 to groups with fixed order of $1,2,3$. This yields

$$\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0} \Big[(1,0,2,3) + (1,2,0,3) + (2,1,0,3) + (1,2,3,0) + (2,1,3,0) + (2,3,1,0) + 2 \leftrightarrow 3\Big] \tag{82}$$

and analogous terms proportional to $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_2}{h_0}$ and $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_3}{h_0}$.

Step 2: Applying Lemma 2, the above coefficients of $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0}$ can be rewritten as

$$\begin{aligned}
(1,0,2,3) &= \frac{1}{h_1}\,([0,1],2,3) \\
(1,2,0,3) + (2,1,0,3) &= \frac{\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2}{h_1}\,(2,1,0,3) + \frac{1}{h_1}\,(2,[0,1],3) \\
(1,2,3,0) + (2,1,3,0) + (2,3,1,0) &= \frac{\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2}{h_1}\,(2,1,3,0) + \frac{\tilde{\mathbf{q}}_1 \cdot (\tilde{\mathbf{q}}_2 + \tilde{\mathbf{q}}_3)}{h_1}\,(2,3,1,0) \\
&\quad + \frac{1}{h_1}\,(2,3,[0,1])
\end{aligned} \tag{83}$$

and analogously for permutations $2 \leftrightarrow 3$. Applying Lemma 1 to the "boundary terms" yields

$$\begin{aligned}
\frac{1}{h_1}\Big[([0,1],2,3) + (2,[0,1],3) + (2,3,[0,1])\Big] \quad &= \\
\frac{(\tilde{\mathbf{q}}_0 + \tilde{\mathbf{q}}_1) \cdot \tilde{\mathbf{q}}_2}{h_1 h_{0,1}}\,(2,[0,1],3) \quad &+ \quad \frac{(\tilde{\mathbf{q}}_0 + \tilde{\mathbf{q}}_1) \cdot (\tilde{\mathbf{q}}_2 + \tilde{\mathbf{q}}_3)}{h_1 h_{0,1}}\,(2,3,[0,1])
\end{aligned} \tag{84}$$

and analogously for permutations $2 \leftrightarrow 3$. Hence, the coefficient of $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0}$ becomes

$$\begin{aligned}
&\frac{\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2}{h_1}\Big[(2,1,0,3) + (2,1,3,0) + (2,3,1,0) + (3,2,1,0)\Big] \\
+ \quad &\frac{(\tilde{\mathbf{q}}_0 + \tilde{\mathbf{q}}_1) \cdot \tilde{\mathbf{q}}_2}{h_1 h_{0,1}}\Big[(2,[0,1],3) + (2,3,[0,1]) + (3,2,[0,1])\Big] \quad + \quad 2 \leftrightarrow 3
\end{aligned} \tag{85}$$

The coefficients of $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_2}{h_0}$ and $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_3}{h_0}$ can be rewritten similarly by applying Lemma 2 to groups with 2 and 3 running from the left end to the position of 0 with the order of all other variables fixed.

Step 3: We finally show, as an example, how another scalar product of momenta can be extracted from the coefficient $(2,1,0,3)+(2,1,3,0)+(2,3,1,0)+(3,2,1,0)$ of $\frac{\tilde{\mathbf{q}}_0 \cdot \tilde{\mathbf{q}}_1}{h_0} \frac{\tilde{\mathbf{q}}_1 \cdot \tilde{\mathbf{q}}_2}{h_1}$, obtained in the second step above. Lemma 2 yields

$$\begin{aligned}
(2,1,0,3) &= \frac{1}{h_2}\,([1,2],0,3) \\
(2,1,3,0) &= \frac{1}{h_2}\,([1,2],3,0) \\
(2,3,1,0)+(3,2,1,0) &= \frac{\tilde{\mathbf{q}}_2 \cdot \tilde{\mathbf{q}}_3}{h_2}\,(3,2,1,0) + \frac{1}{h_2}\,(3,[1,2],0)
\end{aligned} \tag{86}$$

Applying Lemma 2 to the boundary terms in (86) yields

$$\begin{aligned}
\frac{1}{h_2}\,([1,2],0,3) &= \frac{1}{h_2 h_{1,2}}\,([0,1,2],3) \\
\frac{1}{h_2}\Big[([1,2],3,0)+(3,[1,2],0)\Big] &= \frac{(\tilde{\mathbf{q}}_1+\tilde{\mathbf{q}}_2)\cdot\tilde{\mathbf{q}}_3}{h_2 h_{1,2}}\,(3,[1,2],0) + \frac{1}{h_2 h_{1,2}}\,(3,[0,1,2])
\end{aligned} \tag{87}$$

For the boundary terms above, Lemma 1 yields finally

$$\frac{1}{h_2 h_{1,2}}\Big[([0,1,2],3)+(3,[0,1,2])\Big] = \frac{(\tilde{\mathbf{q}}_0+\tilde{\mathbf{q}}_1+\tilde{\mathbf{q}}_2)\cdot\tilde{\mathbf{q}}_3}{h_2 h_{1,2} h_{0,1,2} h_3} \tag{88}$$

Thus only terms which contain products of three scalar products of momenta are left over.

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